A Comparison of Some Linear Quadratic Gaussian Discrete Control Algorithms

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Abstract

The paper presents a new discrete linear quadratic Gaussian (LQG) control algorithm and compares it with three existing optimal discrete LQG control algorithms. The control system is a single-input-single-output (SISO) regulating control system, and the model is an ARMAX model. The new algorithm gives the same controller as that of the three existing algorithms. However, the new control algorithm is easy to understand and more versatile. It can be used to design other stochastic control models such as the Box-Jenkins and innovation state space models and also deterministic linear quadratic tracking controllers.

Keywords: ARMAX model, Box-Jenkins model, linear quadratic Gausian, state space model.

1 Introduction

Since the publication of his book, K. Vu (2008), the author of this paper received numerous emails asking about an LQG discrete control algorithm in the book. Most of the emails came from graduate students in control and process control engineering, who wanted a comparison of the control algorithm with other algorithms established in popular textbooks. Also, since the author used the Box-Jenkins stochastic control model, G. Box and G. Jenkins (1976), the request wanted the equivalent algorithm for the ARMAX model. The author called the algorithm the infinite steps optimal control algorithm. It is one of the four essential discrete control algorithms that every graduate student in control engineering or working control engineer should know. As a response from this request, the author will present a new optimal LQG control algorithm, in this paper, and make a comparison of this control algorithm with three other popular control algorithms discussed in good textbooks. The paper is organized as follows. Section one is the introduction section. Section two describes three popular LQG stochastic control algorithms. In section three, the new algorithm is described. In section four, the algorithms are compared and section five concludes the discussion of the paper.

2 The Optimal LQG Control Algorithms

In this section, three well-known discrete LQG control algorithms will be discussed. The algorithms are extracted from popular textbooks. The algorithms are credited to the authors of the book. This does not necessarily mean that they are the originators of the algorithms, but it means there is a lack of more resources from the author to give credits. Also, the symbols in the algorithms are retained as they are written in the algorithms as much as possible to create less confusion when a reader wants to consult the algorithms.

2.1 The Algorithm of K. J. Åström and B. Wittenmark

The model used in K.J. Åström and B. Wittenmark (1990) has a different look from the standard model of other algorithms discussed in this paper, but it is indeed an ARMAX model. The model for this algorithm is

$$A(q)y(k) = B(q)u(k) + C(q)e(k)$$

where q is the forward-shift operator. The performance index for the algorithm is

$$J_{lq} = Min\ E\{y(k)^2 + \rho u(k)^2\}.$$

The algorithm can be described as follows.

- Obtain the polynomial $P(z)$ from the following spectral factorization equation
$$rP(z)P(z^{-1}) = \rho A(z)A(z^{-1}) + B(z)B(z^{-1}).$$
- Solve the following Diophantine equation
$$A(z)R(z) + B(z)S(z) = P(z)C(z) \quad (1)$$
for the polynomials $R(z)$ and $S(z)$.
- Obtain the controller as
$$u(k) = -\frac{S(q)}{R(q)}y(k).$$

2.2 The Algorithm of M. J. Grimble

The algorithm first appears in the self-tuning control context in M.J. Grimble and M.A. Johnson (1988). It is then discussed as a time-invariant controller in M.J. Grimble (1994). The model used is

$$\begin{aligned} A(z^{-1})y_t &= B(z^{-1})u_t + C(z^{-1})\varepsilon_t, \\ e_t &= y_t + v_t \end{aligned} \quad (2)$$

with

$$\begin{aligned} A(z^{-1}) &= 1 + a_1 z^{-1} + \cdots, \\ B(z^{-1}) &= (b_0 + b_1 z^{-1} + \cdots)z^{-k}, \\ C(z^{-1}) &= 1 + c_1 z^{-1} + \cdots. \end{aligned}$$

The performance criterion to be minimized is defined as

$$J = E\{Q_c e_t^2 + R_c u_t^2\}.$$

The algorithm can be described as follows.

- The algorithm first looks for the polynomial D_f from the spectral equation

$$D_f D_f^* = CQ_2C^* + ARA^* \tag{3}$$

  where Q_2 and R are white disturbance noise ε_t and measurement noise v_t variances.

- Then it looks for the polynomial D_c from

$$D_c^* D_c = B^* Q_c B + A^* R_c A.$$

- The polynomials of the feedback controller are obtained from the following Diophantine equation

$$AH_0 + BG_0 = D_c D_f. \tag{4}$$

- The feedback controller is

$$C_0 = \frac{G_0}{H_0}.$$

2.3 The Algorithm of V. Kučera

The algorithm of V. Kučera (1979) is designed for multivariable systems, but it can also be used for an SISO system. The algorithm is interpreted as follows.

The model of the system is

$$\begin{aligned} y &= Su + Qw, \\ &= A^{-1}Bu + A^{-1}Cw, \\ e &= y + v \end{aligned}$$

with the feedback controller

$$u = -Re.$$

The control criterion of the algorithm is the minimum of

$$v = tr < \Phi W_u > + tr < \Psi W_y >$$

where W_u and W_y are variances of the variables u and y and Φ and Ψ are their weighting constants.

The design procedure can be described as follows.

- Obtain the polynomials A_1, B_1, A_2 and B_2 from the following equations:

$$\begin{aligned} S &= A_1^{-1}B_1, \\ &= B_2 A_2^{-1}. \end{aligned}$$

  For an SISO system, $A_1 = A_2$ and $B_1 = B_2$; but they are not necessarily the same as A and B.

- Obtain two polynomials D and E from the following two spectral factorization equations:

$$\begin{aligned} A\Phi_1 A_* + C\Psi_1 C_* &= DD_*, \\ A_{2*}\Phi A_2 + B_{2*}\Psi B_2 &= E_* E. \end{aligned}$$

  The parameters Φ_1 and Ψ_1 are the variances of the white noises v and w.

- Create four polynomials A_3, D_2, B_3 and D_1 from the following equations:

$$\begin{aligned} D^{-1}A &= A_3 D_2^{-1}, \\ D^{-1}B &= B_3 D_1^{-1}. \end{aligned}$$

- Obtain three minimum-degree polynomials X, Y and Z from the following Diophantine equations:

$$\begin{aligned} \bar{E}X - ZB_3 &= \bar{A}_2 \Phi D_1, \\ \bar{E}Y + ZA_3 &= \bar{B}_2 \Psi D_2 \end{aligned}$$

  with $\bar{A}_2 = d^p A_{2*}$, $\bar{B}_2 = d^p B_{2*}$ and $\bar{E} = d^p E_*$ where d is the backward shift operator.

- Obtain the controller as

$$R = D_1 X^{-1} Y D_2^{-1}.$$

The minimal cost or performance index value is

$$\begin{aligned} v_{min} = & -tr < \Phi_1 \Psi > \\ & + tr < D_* A_*^{-1} \Psi A^{-1} D \\ & \quad - D_* A_*^{-1} \Psi B_2 E^{-1} E_*^{-1} B_{2*} \Psi A^{-1} D > \\ & + tr < Z_* E^{-1} E_*^{-1} Z > . \end{aligned}$$

3 The New Algorithm

The model of the control system is given by Eq. (2) as

$$A(z^{-1})y_t = B(z^{-1})u_{t-k} + C(z^{-1})\varepsilon_t.$$

With the performance index

$$Min\, \sigma^2 = Min\, \sigma_y^2 + \lambda \sigma_u^2$$

where σ_y^2 and σ_u^2 are the variances of the output and input variables under feedback, the new algorithm gives the controller as below

$$\begin{aligned} u_t &= -\frac{A(z^{-1})E(z^{-1})}{C(z^{-1})D(z^{-1}) - B(z^{-1})E(z^{-1})z^{-k}} y_t, \\ &= -\frac{A(z^{-1})E(z^{-1})}{A(z^{-1})G(z^{-1})} y_t, \\ &= -\frac{E(z^{-1})}{G(z^{-1})} y_t \end{aligned} \tag{5}$$

where the new polynomials satisfy the following equations:

$$D(z)D(z^{-1}) = B(z)B(z^{-1}) + \lambda A(z)A(z^{-1}), \quad (6)$$

$$\frac{C(z^{-1})}{A(z^{-1})} = \psi(z^{-1}) + \frac{\gamma(z^{-1})}{A(z^{-1})} z^{-k}, \quad (7)$$

$$\frac{\gamma(z^{-1})B(z)}{A(z^{-1})D(z)} = \frac{E(z^{-1})}{A(z^{-1})} + \frac{F(z)}{D(z)} z. \quad (8)$$

Furthermore, the optimal performance index is

$$Min\ \sigma^2 = \sigma_\varepsilon^2 \underset{z=0}{Residue} \frac{\psi(z)\psi(z^{-1})}{z} + \lambda \frac{\gamma(z)\gamma(z^{-1})}{zD(z)D(z^{-1})} + \frac{F(z)F(z^{-1})}{zD(z)D(z^{-1})}$$

where σ_ε^2 is the variance of the white noise ε_t.

4 Comparison and Discussion

First, an example is given to verify the new algorithm. For this purpose, consider the following control system with the parameters:

$$\begin{aligned} A(z^{-1}) &= 1 - 0.9426z^{-1} + 0.9695z^{-2}, \\ B(z^{-1}) &= 0.4451 + 0.3761z^{-1}, \\ C(z^{-1}) &= 1 + 0.1827z^{-1} + 0.0680z^{-2}, \end{aligned}$$

the dead time $k = 2$ and the white noise of unit variance. The design procedure gives the following polynomials:

$$\begin{aligned} D(z^{-1}) &= 0.6828 + 0.0989z^{-1} + 0.0710z^{-2}, \\ E(z^{-1}) &= -0.1671 - 0.9294z^{-1}, \\ F(z^{-1}) &= 0.1536 + 0.0119z^{-1}. \end{aligned}$$

The controller is

$$u_t = -\frac{-0.2448 - 1.3612z^{-2}}{1 + 1.2703z^{-1} + 0.5354z^{-2}} y_t.$$

The performance index value can be obtained as follows

$$\begin{aligned} \hat{\sigma}^2 &= \underset{z=0}{Residue} \\ &\quad [\frac{\psi(z)\psi(z^{-1})}{z} + \lambda \frac{\gamma(z)\gamma(z^{-1})}{zD(z)D(z^{-1})} + \frac{F(z)F(z^{-1})}{zD(z)D(z^{-1})}], \\ &= 2.2664 + 0.05 \times 2.7833 + 0.0513, \\ &= 2.4569, \\ &= 2.3631 + 0.05 \times 1.8777, \\ &= \underset{z=0}{Residue} [\frac{G(z)G(z^{-1})}{zD(z)D(z^{-1})} + 0.05 \frac{E(z)E(z^{-1})}{zD(z)D(z^{-1})}], \\ &= Min\ \sigma_y^2 + \lambda \sigma_u^2. \end{aligned}$$

The match of the variances implies that the control algorithm is correct. Now, the results of some examples of the three existing algorithms will be compared with that of the new algorithm.

Reference K.J. Åström and B. Wittenmark (1990) does give a good example, so a quick check for this algorithm can be established quite easily. Example 12.7 on page 393 gives the model

$$\begin{aligned} A(z) &= z + a, \\ B(z) &= b, \\ C(z) &= z + c. \end{aligned}$$

Now, let the parameters have the values $a = -0.25$, $b = 2$, $c = -0.5$ and the penalty constant $\rho = 0.05$. With these values, the following parameters can be obtained.

$$\begin{aligned} r &= \frac{\rho(1+a^2) + b^2 + \sqrt{\rho^2(1-a^2)^2 + 2\rho b^2(1+a^2) + b^4}}{2}, \\ &= 4.053086449 \end{aligned}$$

and

$$\begin{aligned} p_1 &= \rho a / r, \\ &= -0.003084069, \\ p_1 c/a &= -0.0061681388. \end{aligned}$$

The obtained controller, on page 394, is

$$\begin{aligned} u(k) &= -\frac{b(c-a)}{r(1-ap_1)} \frac{q}{q + p_1 c/a} y(k), \\ &= -\frac{-0.123457965}{1 - 0.006181388z^{-1}} y(k). \end{aligned}$$

For the new algorithm, the model of the system is

$$\begin{aligned} A(z^{-1}) &= 1 - 0.25z^{-1}, \\ B(z^{-1}) &= 2, \\ C(z^{-1}) &= 1 - 0.5z^{-1} \end{aligned}$$

with the dead time $k = 1$. The new algorithm gives the following results:

$$\begin{aligned} A(z^{-1}) &= 1 - 0.25z^{-1}, \\ B(z^{-1}) &= 2, \\ C(z^{-1}) &= 1 - 0.5z^{-1}, \\ D(z^{-1}) &= 2.013227868 - 0.0062089z^{-1}, \\ E(z^{-1}) &= -0.248549, \\ F(z^{-1}) &= -0.0015432245. \end{aligned}$$

The obtained feedback controller is

$$u_t = -\frac{-0.123457965}{1 - 0.0061681388z^{-1}} y_t.$$

The performance index value has the value

$$\begin{aligned} \hat{\sigma}^2 &= \underset{z=0}{Residue} \\ &\quad [\frac{\psi(z)\psi(z^{-1})}{z} + \rho \frac{\gamma(z)\gamma(z^{-1})}{zD(z)D(z^{-1})} + \frac{F(z)F(z^{-1})}{zD(z)D(z^{-1})}], \\ &= 1 + 0.05 \times 0.01542 + 5.8759 \times 10^{-7}, \end{aligned}$$

$$
\begin{aligned}
&= 1.0007716,\\
&= 1.0000095 + 0.05 \times 0.015242,\\
&= \underset{z=0}{Residue}\,[\frac{G(z)G(z^{-1})}{zD(z)D(z^{-1})} + 0.05\frac{E(z)E(z^{-1})}{zD(z)D(z^{-1})}],\\
&= Min\ \sigma_y^2 + \lambda\sigma_u^2.
\end{aligned}
$$

The algorithm of K.J. Åström and B. Wittenmark (1990) gives the same result as the result of the new algorithm, for this example. More tests show that the two algorithms give the same results. The tests are based on the following equation

$$
A(z^{-1})G(z^{-1}) + B(z^{-1})E(z^{-1})z^{-k} = D_n(z^{-1})C(z^{-1})
$$

where $D_n(z^{-1})$ is a monic polynomial derived from the polynomial $D(z^{-1})$ by dividing every coefficient of this polynomial by its first coefficient with no power. The last equation is actually Eq. (1) of the algorithm of K.J. Åström and B. Wittenmark (1990).

The algorithm of M.J. Grimble (1994) uses an orthodox model with the backward shift operator. It provides an additional noise to the output variable. Reference M.J. Grimble (1994) does give a good example for verification of the algorithm.

The example on page 69 of reference M.J. Grimble (1994) gives a system with the following model

$$
\begin{aligned}
A(z^{-1}) &= 1 - 0.7z^{-1},\\
B(z^{-1}) &= z^{-3},\\
C(z^{-1}) &= 1
\end{aligned}
$$

with no measurement noise and the weighting constants $Q_c = 1, R_c = 0.1$. The reference gives the controller

$$
\frac{G_0}{H_0} = -\frac{0.313036}{1 + 0.63885z^{-1} + 0.447195z^{-2}}.
$$

A minus sign must be added, which is contrary to what is reported in the reference because the signal is negated as shown in the block diagram on page 28. The new algorithm gives the following matching results.

$$
\begin{aligned}
D(z^{-1}) &= 1.0699156 - 0.0654257z^{-1},\\
E(z^{-1}) &= 0.334922462,\\
F(z^{-1}) &= 0.021912543
\end{aligned}
$$

with the controller

$$
u_t = -\frac{0.313036}{1 + 0.63885z^{-1} + 0.447195z^{-2}}y_t
$$

and the following performance index value

$$
\begin{aligned}
\hat{\sigma}^2 &= \underset{z=0}{Residue}\\
&\quad [\frac{\psi(z)\psi(z^{-1})}{z} + \rho\frac{\gamma(z)\gamma(z^{-1})}{zD(z)D(z^{-1})} + \frac{F(z)F(z^{-1})}{zD(z)D(z^{-1})}],\\
&= 1.7301 + 0.1 \times 0.103161 + 0.000421,\\
&= 1.7408371,\\
&= 1.73100119. + 0.1 \times 0.098359544,\\
&= \underset{z=0}{Residue}\,[\frac{G(z)G(z^{-1})}{zD(z)D(z^{-1})} + 0.05\frac{E(z)E(z^{-1})}{zD(z)D(z^{-1})}],\\
&= Min\ \sigma_y^2 + \lambda\sigma_u^2.
\end{aligned}
$$

It can be expected that this algorithm will give the same result as that of the new algorithm for the case of no measurement noise. This is because Eq. (4) will be the same as Eq. (1) in the algorithm of K.J. Åström and B. Wittenmark (1990) if, in Eq. (3), $R = 0$ and $Q_2 = 1$.

For an SISO system, the algorithm in V. Kučera (1979) will reduce to the algorithm in M.J. Grimble (1994). Reference V. Kučera (1979) does not give a good example for a quick verification, so the model of the example in K.J. Åström and B. Wittenmark (1990) will be used to verify this algorithm.

The control system, in the symbols of the algorithm of V. Kučera (1979), is

$$
\begin{aligned}
S &= A^{-1}B = \frac{bd}{1+ad} = \frac{2}{1-0.25d},\\
&= A^{-1}C = \frac{1+cd}{1+ad} = \frac{1-0.5d}{1-0.25d}.
\end{aligned}
$$

The letter d is the backward shift operator of the algorithm of V. Kučera (1979). Then, it can be obtained that

$$
A_2 = A_1 = A, B_2 = B_1 = B.
$$

With the values $\phi_1 = 0$ and $\psi_1 = 1$, it can be obtained that $D = C = 1 - 0.5d$. With the values of the weighting constants $\phi = 0$ and $\psi = 1$, it can be obtained that

$$
E = 2.013227868 - 0.0062089d.
$$

Since the polynomials are scalar polynomials with no common divisor, it can be obtained that $A_3 = A$, $D_2 = D$, $B_3 = B$ and $D_1 = D$. The two Diophantine equations can be combined into one as follows. We write

$$
\begin{aligned}
A_3\bar{E}X - ZA_3B_3 &= A_3\bar{A}_2\phi D_1,\\
B_3\bar{E}Y + ZA_3B_3 &= B_3\bar{B}_2\psi D_1.
\end{aligned}
$$

By adding the last two equations together, it can be obtained that

$$
\begin{aligned}
(A_3X + B_3Y)\bar{E} &= (A_3\bar{A}_2\phi + B_3\bar{B}_2\psi)D,\\
(AX + BY)d^pE_* &= d^p(\phi AA_* + \psi BB_*)D,\\
(AX + BY)E_* &= EE_*D,\\
AX + BY &= ED.
\end{aligned}
$$

The last equation is actually the equation to solve for the controller's polynomials in the algorithms of K.J. Åström and B. Wittenmark (1990) and M.J. Grimble (1994). It can, therefore, be said that the controller is

$$
R = \frac{Y}{X} = \frac{-0.123457965}{1 - 0.0061681388d}.
$$

In the above discussion, it has been seen that the algorithms give the same controller for the same control system. The algorithms of K.J. Åström and B. Wittenmark (1990) and M.J. Grimble (1994) are simple algorithms. The latter allows for additional measurement noise. The algorithm of V. Kučera (1979) is more sophisticated and can be used to design multivariable systems. All of these algorithms seem, however, to have one common weakness: They are not easy to understand and the design steps lack physical meaning.

The design procedure of the new algorithm can be summarized as follows.

- The spectral factorization equation, Eq. (6), is solved to obtain the polynomial $D(z^{-1})$ for a chosen balance between minimum variance control and open-loop condition. This step is common to every algorithm.

- The first Diophantine equation, Eq. (7), is solved to separate the controllable and uncontrollable variances of the output variable.

$$\frac{C(z^{-1})}{A(z^{-1})} = \psi(z^{-1}) + \frac{\gamma(z^{-1})}{A(z^{-1})}z^{-k}$$

- The second Diophantine equation, Eq. (8), is solved to separate the past and future of the disturbance values.

$$\frac{\gamma(z^{-1})B(z)}{A(z^{-1})D(z)} = \underbrace{\frac{E(z^{-1})}{A(z^{-1})}}_{\text{past and current disturbance}} - \underbrace{\frac{F(z)}{D(z)}z}_{\text{future disturbance}} .$$

- The controller is obtained as given by Eq. (5).

The new algorithm is easy to understand. Similar algorithms have been used to design controllers for other stochastic control system models such as the Box-Jenkins and the innovation state space models K. Vu (2008). These algorithms were written as Matlab[1] m files: **lqgarmax**, **lqgbj** and **lqgss** in the Discrete Control Toolbox RTF-SISO. The algorithm can also be used to design linear quadratic tracking controllers. Since regulating control is stochastic, the future disturbance is not known; the controller cannot compensate for this part. For deterministic tracking control, the future disturbance is known; the controller can compensate for this future part. The controller is known as the 2.5 degree-of-freedom controller. All of these controllers are discussed in K. Vu (2008).

5 Conclusion

In this paper, a new algorithm to obtain the stochastic controller for an ARMAX model is presented. The algorithm is easy to understand; it is also very versatile because it can be used to design controllers for other stochastic control models. It can also be used to design deterministic tracking controllers. With such versatility, it should be the common algorithm, described in textbooks like other algorithms, to design linear quadratic controllers.

[1]Matlab is a registered trademark of The MathWorks, Inc.

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Appendix

From the model of the control system given by Eq. (2), we can write

$$\begin{aligned} y_{t+k} &= \frac{B(z^{-1})}{A(z^{-1})}u_t + \frac{C(z^{-1})}{A(z^{-1})}\varepsilon_{t+k}, \\ &= \frac{B(z^{-1})}{A(z^{-1})}u_t + \psi(z^{-1})\varepsilon_{t+k} + \frac{\gamma(z^{-1})}{A(z^{-1})}\varepsilon_t. \end{aligned}$$

The control system equation resembles that of a Box-Jenkins model control equation. Therefore, it is possible to use the algorithm described in K. Vu (2008) to derive the controller. Now, let the open-loop controller be of the form

$$u_t \quad = \quad l(z^{-1})\varepsilon_t.$$

Then the output variable y_t of the system under feedback can be written as

$$y_{t+k} \quad = \quad \psi(z^{-1})\varepsilon_{t+k} + \frac{B(z^{-1})l(z^{-1}) + \gamma(z^{-1})}{A(z^{-1})}\varepsilon_t.$$

By using a variance formula described in K. Vu (2007), the variance of the output variable under feedback can be written as

$$\sigma_y^2 = \sigma_\varepsilon^2 \oint_C \big[\psi(z) + \frac{B(z)l(z) + \gamma(z)}{A(z)}z^k\big] \big[\psi(z^{-1}) + \frac{B(z^{-1})l(z^{-1}) + \gamma(z^{-1})}{A(z^{-1})}z^{-k}\big]\frac{dz}{z}$$

with σ_ε^2 as the variance of the white noise ε_t. However, as explained in K. Vu (2007a), the variance must now be evaluated as

$$\sigma_y^2 = \sigma_\varepsilon^2 \underset{z=0}{Residue}\big[\psi(z) + \frac{B(z)l(z) + \gamma(z)}{A(z)}z^k\big] \big[\psi(z^{-1}) + \frac{B(z^{-1})l(z^{-1}) + \gamma(z^{-1})}{A(z^{-1})}z^{-k}\big]\frac{1}{z}.$$

Similarly, the variance of the input variable under feedback can be written as

$$\sigma_u^2 = \sigma_\varepsilon^2 \underset{z=0}{Residue}\frac{l(z)l(z^{-1})}{z}.$$

With the performance index

$$Min\ \sigma^2 \quad = \quad Min\ \sigma_y^2 + \lambda\sigma_u^2,$$

the following equation is true, viz

$$\sigma^2 = \sigma_\varepsilon^2 \underset{z=0}{Residue}\big[\psi(z) + \frac{B(z)l(z) + \gamma(z)}{A(z)}z^k\big] \big[\psi(z^{-1}) + \frac{B(z^{-1})l(z^{-1}) + \gamma(z^{-1})}{A(z^{-1})}z^{-k}\big]\frac{1}{z} + \lambda\frac{l(z)l(z^{-1})}{z}.$$

The first term on the right hand side of the last equation has four components. The residues at the zero pole of the two cross-components are zero because the coefficients of the terms with power -1, i.e. the residues, are missing. Therefore, it can be written as

$$\sigma^2 = \sigma_\varepsilon^2 \underset{z=0}{Residue} \quad \psi(z)\psi(z^{-1})\frac{1}{z} + \big[\frac{B(z)l(z) + \gamma(z)}{A(z)}\big] \big[\frac{B(z^{-1})l(z^{-1}) + \gamma(z^{-1})}{A(z^{-1})}\big]\frac{1}{z} + \lambda\frac{A(z)l(z)l(z^{-1})A(z^{-1})}{zA(z)A(z^{-1})}.$$

Now, by defining the following spectral factorization equation

$$D(z)D(z^{-1}) \quad = \quad B(z)B(z^{-1}) + \lambda A(z)A(z^{-1}),$$

the second last equation can be written as

$$\sigma^2 = \sigma_\varepsilon^2 \underset{z=0}{Residue} \big[\psi(z)\psi(z^{-1})\frac{1}{z} + \frac{l(z)D(z)D(z^{-1})l(z^{-1})}{zA(z)A(z^{-1})} + \frac{B(z)l(z)\gamma(z^{-1}) + B(z^{-1})l(z^{-1})\gamma(z) + \gamma(z)\gamma(z^{-1})}{zA(z)A(z^{-1})}\big]$$

or

$$\sigma^2 = \sigma_\varepsilon^2 \underset{z=0}{Residue}\frac{\psi(z)\psi(z^{-1})}{z} + \big[\frac{l(z)D(z)}{A(z)} + \frac{\gamma(z)B(z^{-1})}{A(z)D(z^{-1})}\big] \big[\frac{l(z^{-1})D(z^{-1})}{A(z^{-1})} + \frac{\gamma(z^{-1})B(z)}{A(z^{-1})D(z)}\big]\frac{1}{z} - \frac{B(z)B(z^{-1})\gamma(z)\gamma(z^{-1})}{zD(z)D(z^{-1})A(z)A(z^{-1})} + \frac{\gamma(z)\gamma(z^{-1})}{zA(z)A(z^{-1})}.$$

The last two terms of the last equation can be joined together and the equation can be written as

$$\sigma^2 = \sigma_\varepsilon^2 \underset{z=0}{Residue}\frac{\psi(z)\psi(z^{-1})}{z} + \lambda\frac{\gamma(z)\gamma(z^{-1})}{zD(z)D(z^{-1})} + \big[\frac{l(z)D(z)}{A(z)} + \frac{\gamma(z)B(z^{-1})}{A(z)D(z^{-1})}\big] \big[\frac{l(z^{-1})D(z^{-1})}{A(z^{-1})} + \frac{\gamma(z^{-1})B(z)}{A(z^{-1})D(z)}\big]\frac{1}{z}.$$

Now, by defining the following spectral separation equation

$$\frac{\gamma(z^{-1})B(z)}{A(z^{-1})D(z)} \quad = \quad \frac{E(z^{-1})}{A(z^{-1})} + \frac{F(z)}{D(z)}z,$$

the second last equation can be written as

$$\sigma^2 = \sigma_\varepsilon^2 \underset{z=0}{Residue} \frac{\psi(z)\psi(z^{-1})}{z} + \lambda\frac{\gamma(z)\gamma(z^{-1})}{zD(z)D(z^{-1})} + \frac{F(z)F(z^{-1})}{zD(z)D(z^{-1})} + \big[\frac{l(z)D(z)}{A(z)} + \frac{E(z)}{A(z)}\big]\big[\frac{l(z^{-1})D(z^{-1})}{A(z^{-1})} + \frac{E(z^{-1})}{A(z^{-1})}\big]\frac{1}{z}.$$

The performance index value has four non-negative terms. Only the last one depends on the controller. The performance index value, therefore, obtains its minimal value when the last term is zero. This means that the controller is

$$l(z^{-1}) \quad = \quad -\frac{E(z^{-1})}{D(z^{-1})}.$$

With this controller, the performance index has the following minimal value

$$\hat{\sigma}^2 = \sigma_\varepsilon^2 \underset{z=0}{Residue} \frac{\psi(z)\psi(z^{-1})}{z} + \lambda\frac{\gamma(z)\gamma(z^{-1})}{zD(z)D(z^{-1})} + \frac{F(z)F(z^{-1})}{zD(z)D(z^{-1})}.$$

Under feedback, the input and output variables time series are:

$$
\begin{aligned}
u_t &= -\frac{E(z^{-1})}{D(z^{-1})}\varepsilon_t, \\
y_t &= -\frac{B(z^{-1})}{A(z^{-1})}\frac{E(z^{-1})z^{-k}}{D(z^{-1})}\varepsilon_t + \frac{C(z^{-1})}{A(z^{-1})}\varepsilon_t, \\
&= \frac{C(z^{-1})D(z^{-1}) - B(z^{-1})E(z^{-1})z^{-k}}{A(z^{-1})D(z^{-1})}\varepsilon_t.
\end{aligned}
$$

From these two time series, the feedback controller in its implementable form can be obtained as

$$
\begin{aligned}
u_t &= -\frac{A(z^{-1})E(z^{-1})}{C(z^{-1})D(z^{-1}) - B(z^{-1})E(z^{-1})z^{-k}}y_t, \\
&= -\frac{A(z^{-1})E(z^{-1})}{A(z^{-1})G(z^{-1})}y_t, \\
&= -\frac{E(z^{-1})}{G(z^{-1})}y_t.
\end{aligned}
$$

In the above equations, the greatest common polynomial $A(z^{-1})$ of the numerator and denominator is canceled out. The denominator must necessarily contain this factor as this fact can be proved as follows.

From the two Diophantine equations, the following equations are true, viz

$$
\begin{aligned}
C(z^{-1})D(z^{-1}) &= \psi(z^{-1})A(z^{-1})D(z^{-1}) \\
&\quad +\gamma(z^{-1})D(z^{-1})z^{-k}, \\
B(z^{-1})E(z^{-1})z^{-k} &= B(z^{-1}) \\
&\quad [\frac{\gamma(z^{-1})B(z) - A(z^{-1})F(z)z}{D(z)}]z^{-k}.
\end{aligned}
$$

Then, it can be written as follows.

$$
\begin{aligned}
&C(z^{-1})D(z^{-1}) - B(z^{-1})E(z^{-1})z^{-k} = \\
&\quad \psi(z^{-1})A(z^{-1})D(z^{-1}) + \\
&\quad [\frac{\gamma(z^{-1})D(z^{-1})D(z) - \gamma(z^{-1})B(z^{-1})B(z)}{D(z)} + \\
&\qquad \frac{A(z^{-1})B(z^{-1})F(z)z}{D(z)}]z^{-k} = \\
&\quad \psi(z^{-1})A(z^{-1})D(z^{-1}) + \\
&[\frac{\lambda\gamma(z^{-1})A(z^{-1})A(z) + A(z^{-1})B(z^{-1})F(z)z}{D(z)}]z^{-k} = \\
&\qquad A(z^{-1}) \times \\
&\quad [\psi(z^{-1})D(z^{-1}) + \frac{\lambda\gamma(z^{-1})A(z) + B(z^{-1})F(z)z}{D(z)}]z^{-k}.
\end{aligned}
$$

The right hand side of the last equation contains the factor $A(z^{-1})$, so the left hand side must also contains it.