Four Essential Optimal Discrete Controllers for Control Applications

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Abstract

Four optimal discrete controllers for a single-input-single-output (SISO), state space model, stochastic regulating control system are presented. They are the one step optimal or myopic controller, the N steps optimal controller, the pseudo-infinite steps optimal controller and the infinite steps optimal controller. Each controller has different characteristics. Depending on the application, a particular controller might be stronger than the others and be the most suitable controller for the application.

Keywords: Kalman filter, linear quadratic control, optimal control, state estimator, time series.

1 Introduction

Control theory has applications in many fields of human life. This fact might lead to a situation that it is necessary to have many different controllers for diverse applications. It is, however, unfortunate that academic courses in universities do not provide this necessity. Control engineers working in different industries are, then, often have to design their own controllers for their applications. It is a fact that control algorithm like model predictive control is only popular among chemical process control engineers and control algorithm like sliding mode control seems to be mastered by only electrical control engineers. It is, however, agreed by many professors that a basic knowledge of linear quadratic control theory is essential for all control engineers. Therefore, in this paper, we will explore different algorithms of the linear quadratic control theory. The paper is organized as follows. Section one is the introduction section. The algorithms are discussed in section two. In section three, an example is given. Section four concludes the discussion of the paper.

2 The Optimal Control Algorithms

To begin our discussion, we consider an SISO, state space model, stochastic regulating control system described by the following equation

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{A}\mathbf{x}_t + \mathbf{b}u_{t-f} + \mathbf{w}_t, \\ y_t &= \mathbf{c}\mathbf{x}_t + v_t \end{aligned}$$

where $\mathbf{A}$ is the state transition matrix, $\mathbf{b}$ is the control or input vector, $\mathbf{c}$ is the output or measurement vector and $\mathbf{w}_t$ and v_t are white noise quantities. The vector $\mathbf{w}_t$ is the process noise vector and v_t is the scalar measurement noise. These white noises are not correlated, and their variances are $\mathbf{R}_w$ and σ_v^2.

2.1 The One Step Optimal Controller

The control criterion or performance index for the one step optimal controller is

$$\underset{u_t}{Min} \quad E\{y_{t+f+1}^2 + \lambda u_t^2 \mid \mathcal{Y}_t\}, \qquad \lambda \geq 0.$$

The condition here is *given* $\mathcal{Y}_t$, and $\mathcal{Y}_t$ is the vector that contains all the available output variable values up to the time t. By replacing the output variable y_{t+f+1} in the performance index, we obtain

$$\begin{aligned} &\underset{u_t}{Min} \quad E\{y_{t+f+1}^2 + \lambda u_t^2 \mid \mathcal{Y}_t\} \\ &= \underset{u_t}{Min} \quad E\{(\mathbf{c}\mathbf{x}_{t+f+1} + v_{t+f+1})^2 + \lambda u_t^2 \mid \mathcal{Y}_t\}, \\ &= \underset{u_t}{Min} \quad E\{(\mathbf{c}\mathbf{x}_{t+f+1})^2 + 2\mathbf{c}\mathbf{x}_{t+f+1}v_{t+f+1} + v_{t+f+1}^2 \\ &\qquad + \lambda u_t^2 \mid \mathcal{Y}_t\}. \end{aligned}$$

Since the state variable vector $\mathbf{x}_{t+f+1}$ contains the variable u_t, we have to bring this variable out by replacing the state vector

$$\mathbf{x}_{t+f+1} = \mathbf{A}\mathbf{x}_{t+f} + \mathbf{b}u_t + \mathbf{w}_{t+f}$$

in the performance index to obtain

$$\begin{aligned} &\underset{u_t}{Min} \quad E\{y_{t+f+1}^2 + \lambda u_t^2 \mid \mathcal{Y}_t\} = \\ &\underset{u_t}{Min} \quad E\{\mathbf{x}_{t+f}^T\mathbf{A}^T\mathbf{c}^T\mathbf{c}\mathbf{A}\mathbf{x}_{t+f} + 2\mathbf{x}_{t+f}^T\mathbf{A}^T\mathbf{c}^T\mathbf{c}\mathbf{b}u_t \\ &\quad + \mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{b}u_t^2 + \lambda u_t^2 + 2\mathbf{x}_{t+f}^T\mathbf{A}^T\mathbf{c}^T\mathbf{c}\mathbf{w}_{t+f} \\ &\quad + 2\mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{w}_{t+f}u_t + 2\mathbf{c}\mathbf{x}_{t+f+1}v_{t+f+1} + v_{t+f+1}^2 \\ &\quad + tr(\mathbf{c}^T\mathbf{c}\mathbf{w}_{t+f}\mathbf{w}_{t+f}^T) \mid \mathcal{Y}_t\}. \end{aligned}$$

By taking the expectation, we can write the above equation as

$$\begin{aligned} &\underset{u_t}{Min} \quad E\{y_{t+f+1}^2 + \lambda u_t^2 \mid \mathcal{Y}_t\} = \underset{u_t}{Min} \quad \hat{\mathbf{x}}_{t+f|t}^T\mathbf{A}^T\mathbf{c}^T\mathbf{c}\mathbf{A}\hat{\mathbf{x}}_{t+f|t} \\ &\quad + [\lambda + \mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{b}]u_t^2 + 2\hat{\mathbf{x}}_{t+f|t}^T\mathbf{A}^T\mathbf{c}^T\mathbf{c}\mathbf{b}u_t \\ &\quad + \sigma_v^2 + tr(\mathbf{A}^{f+1^T}\mathbf{c}^T\mathbf{c}\mathbf{A}^{f+1}\mathbf{P}_{t|t}) + tr(\sum_{i=0}^{f}\mathbf{A}^{i^T}\mathbf{c}^T\mathbf{c}\mathbf{A}^i\mathbf{R}_w). \end{aligned}$$

The matrix $\mathbf{P}_{t|t}$ is the steady state variance matrix of the error vector in the estimation of the conditional simultaneous state estimator $\hat{\mathbf{x}}_{t|t}$. The one step optimal control is the fundamental control law. It establishes what has been referred to as the *separation theorem* or the *certainty equivalence principle* (K.J. Åström (1970)). The control law has two tasks. The first task is to take the expectation of the stochastic quantity. The second task is to seek its minimum value. These two tasks can be obtained separately – hence the name *separation theorem*. The first task produces the best state estimator $\hat{\mathbf{x}}_{t+f|t}$ via Kalman filtering. This state estimator is given by the following equation

$$\begin{aligned}\hat{\mathbf{x}}_{t+f|t} &= \mathbf{A}^f\hat{\mathbf{x}}_{t|t} + \sum_{i=1}^{f}\mathbf{A}^{i-1}\mathbf{b}u_{t-i},\\ &= \mathbf{A}^{f-1}\hat{\mathbf{x}}_{t+1|t} + \sum_{i=1}^{f-1}\mathbf{A}^{i-1}\mathbf{b}u_{t-i}.\end{aligned}$$

The second task is the procurement of the linear feedback control law, which gives the control action as a linear function of the estimated state vector.

$$u_t = -[\lambda + \mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{b}]^{-1}\mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{A}\hat{\mathbf{x}}_{t+f|t}.$$

The performance index has the value

$$\begin{aligned}&\underset{u_t}{Min}\quad E\{y_{t+f+1}^2 + \lambda u_t^2 \mid \mathcal{Y}_t\} = \\ &\hat{\mathbf{x}}_{t+f|t}^T\left[\mathbf{A}^T\mathbf{c}^T\mathbf{c}\mathbf{A} - \frac{\mathbf{A}^T\mathbf{c}^T\mathbf{c}\mathbf{b}\mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{A}}{\lambda + \mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{b}}\right]\hat{\mathbf{x}}_{t+f|t}\\ &+\sigma_v^2 + tr(\mathbf{A}^{f+1^T}\mathbf{c}^T\mathbf{c}\mathbf{A}^{f+1}\mathbf{P}_{t|t}) + tr(\sum_{i=0}^{f}\mathbf{A}^{i^T}\mathbf{c}^T\mathbf{c}\mathbf{A}^i\mathbf{R}_w).\end{aligned}$$

The one step optimal controller sees and controls only one point at a time. It is, therefore, short-sighted and earns the name as the *myopic controller*. It is a controller that is vulnerable to nonminimum phase systems. However, due to this nature, there is a relation among the input and output variables with the control penalty constant λ under feedback. This gives a strength in system performance monitoring as one can obtain valuable control statistics to verify system and controller models.

2.2 The N Steps Optimal Controller

The one step optimal control law is used to obtain the N steps optimal control law. This control law has the following performance index

$$\underset{u_t}{Min}\quad E\{\sum_{t=1}^{N} y_{t+f+1}^2 + \lambda u_t^2 \mid \mathcal{Y}_t\}, \qquad \lambda \geq 0.$$

The choice and solution of the problem is N control actions u_t's ($t = 1, \cdots N$). The controllers for these N control actions can be obtained from dynamic programming. The solution for the control law, solved in K. Vu (2008), is given as follows.

At the time t, we calculate the control action u_t as

$$\begin{aligned}u_t &= -[\lambda + \mathbf{b}^T(\mathbf{S}_{t+1} + \mathbf{c}^T\mathbf{c})\mathbf{b}]^{-1}\mathbf{b}^T(\mathbf{S}_{t+1} + \mathbf{c}^T\mathbf{c})\mathbf{A}\hat{\mathbf{x}}_{t+f|t},\\ &= \mathbf{L}_t\hat{\mathbf{x}}_{t+f|t}\end{aligned}$$

where the matrix $\mathbf{S}_{t+1}$ is calculated recursively from the following Riccati equation

$$\mathbf{S}_t = \mathbf{A}^T(\mathbf{S}_{t+1} + \mathbf{c}^T\mathbf{c})\mathbf{A} - \frac{\mathbf{A}^T(\mathbf{S}_{t+1} + \mathbf{c}^T\mathbf{c})\mathbf{b}\mathbf{b}^T(\mathbf{S}_{t+1} + \mathbf{c}^T\mathbf{c})\mathbf{A}}{\lambda + \mathbf{b}^T(\mathbf{S}_{t+1} + \mathbf{c}^T\mathbf{c})\mathbf{b}}$$

with the initial condition

$$\mathbf{S}_N = \mathbf{A}^T\mathbf{c}^T\mathbf{c}\mathbf{A} - \frac{\mathbf{A}^T\mathbf{c}^T\mathbf{c}\mathbf{b}\mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{A}}{\lambda + \mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{b}}.$$

Since only the matrix $\mathbf{S}_{t+1}$ is required in the calculation of u_t, the control algorithm is physically implementable. However, the finite N steps must be known for the calculation of N controllers $\mathbf{L}_t$'s ($t = 1, \cdots N$). The N steps control law is seldom used in engineering endeavor because N is usually not known. It is, however, very applicable in economic control systems, especially where a government-controlled economy exists. A five-year economic plan can be a typical application of this control law.

2.3 The Pseudo Infinite Steps Optimal Controller

If the N finite steps in the N steps optimal control law becomes large, the matrix $\mathbf{S}_t$ will converge to a steady state value given by

$$\mathbf{S}_\infty = \mathbf{A}^T(\mathbf{S}_\infty + \mathbf{c}^T\mathbf{c})\mathbf{A} - \frac{\mathbf{A}^T(\mathbf{S}_\infty + \mathbf{c}^T\mathbf{c})\mathbf{b}\mathbf{b}^T(\mathbf{S}_\infty + \mathbf{c}^T\mathbf{c})\mathbf{A}}{\lambda + \mathbf{b}^T(\mathbf{S}_\infty + \mathbf{c}^T\mathbf{c})\mathbf{b}}.$$

This will happen if the state transition matrix $\mathbf{A}$ has all its eigenvalues inside the unit circle. The control law has a large number of almost identical controllers $\mathbf{L}_t$'s at the beginning of the control period and a few different controllers at the end of the control period. In applications, one usually forgets the last few different controllers and applies only a steady state controller (E. Mosca (1995)) with the control law

$$\begin{aligned}u_t &= \mathbf{L}_\infty\hat{\mathbf{x}}_{t+f|t},\\ &= -[\lambda + \mathbf{b}^T(\mathbf{S}_\infty + \mathbf{c}^T\mathbf{c})\mathbf{b}]^{-1}\mathbf{b}^T(\mathbf{S}_\infty + \mathbf{c}^T\mathbf{c})\mathbf{A}\hat{\mathbf{x}}_{t+f|t}.\end{aligned}$$

In this way, the parameter N is not needed, and the control law can be considered as a fixed control law. This control law can be called the pseudo infinite steps optimal control law. It might not be vulnerable to nonminimum phase systems, but it is still one of the heterodox control laws, due to its derivation, which one can see often enough in control literature.

2.4 The Infinite Steps Optimal Controller

For the infinite steps optimal control law, we convert the control system model to the following innovations state space model

$$\begin{aligned}\hat{\mathbf{x}}_{t+1|t} &= \mathbf{A}\hat{\mathbf{x}}_{t|t-1} + \mathbf{b}u_{t-f} + \mathbf{k}a_t,\\ y_t &= \mathbf{c}\hat{\mathbf{x}}_{t|t-1} + a_t.\end{aligned}$$

The performance index of the control law is

$$\underset{l(z^{-1})}{Min} \quad \sigma_y^2 + \lambda\sigma_u^2$$

where σ_y^2 and σ_u^2 are the variances of the output and input variables and $l(z^{-1})$ is the open loop controller given by the following equation

$$u_t = l(z^{-1})a_t.$$

With the given controller, we can write

$$\begin{aligned}\hat{\mathbf{x}}_{t|t-1} &= [\mathbf{I} - \mathbf{A}z^{-1}]^{-1}[\mathbf{b}l(z^{-1})z^{-f} + \mathbf{k}]a_{t-1},\\ y_t &= (1 + \mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{-1}[\mathbf{b}l(z^{-1})z^{-f} + \mathbf{k}]z^{-1})a_t.\end{aligned}$$

By using the following matrix identity of an inverse

$$[\mathbf{I} - \mathbf{A}z^{-1}]^{-1} = \mathbf{I} + \cdots + (\mathbf{A}z^{-1})^{f-1} + [\mathbf{I} - \mathbf{A}z^{-1}]^{-1}\mathbf{A}^f z^{-f},$$

we can write the output variable time series as

$$\begin{aligned}y_t &= (1 + \sum_{i=1}^{f}\mathbf{c}\mathbf{A}^{i-1}\mathbf{k}z^{-i} + [\mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{-1}\mathbf{A}^f\mathbf{k} +\\ &\qquad \mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{-1}\mathbf{b}l(z^{-1})]z^{-f-1})a_t,\\ &= (1 + \sum_{i=1}^{f}\mathbf{c}\mathbf{A}^{i-1}\mathbf{k}z^{-i})a_t + \mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{-1}\\ &\qquad [\mathbf{b}l(z^{-1}) + \mathbf{A}^f\mathbf{k}]a_{t-f-1}.\end{aligned}$$

The time series is written as a sum of two separate and uncorrelated components. Only the second component depends on the controller, so the unconstrained minimum variance controller can be obtained from this component. To obtain this controller, we set this component to zero, i.e. we have

$$\mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{-1}\mathbf{b}l(z^{-1}) + \mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{-1}\mathbf{A}^f\mathbf{k} = 0.$$

From this equation, we can obtain the minimum variance controller as

$$l_{mv}(z^{-1}) = -\frac{\mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{+}\mathbf{A}^f\mathbf{k}}{\mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{+}\mathbf{b}}.$$

In the general case, the variance of the output variable can be written as a sum of two components as

$$\begin{aligned}\sigma_y^2 &= \frac{\sigma_a^2}{2\pi i}\oint_C (1 + \sum_{i=1}^{f}\mathbf{c}\mathbf{A}^{i-1}\mathbf{k}z^{i})(1 + \sum_{i=1}^{f}\mathbf{c}\mathbf{A}^{i-1}\mathbf{k}z^{-i})\frac{dz}{z} +\\ &\quad \frac{\sigma_a^2}{2\pi i}\oint_C \mathbf{c}[\mathbf{I} - \mathbf{A}z]^{-1}[\mathbf{A}^f\mathbf{k} + \mathbf{b}l(z)]\\ &\qquad \mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{-1}[\mathbf{A}^f\mathbf{k} + \mathbf{b}l(z^{-1})]\frac{dz}{z},\\ &= \sigma_{y,mv}^2 + \frac{\sigma_a^2}{2\pi i}\oint_C \mathbf{c}[\mathbf{I} - \mathbf{A}z]^{-1}[\mathbf{A}^f\mathbf{k} + \mathbf{b}l(z)]\\ &\qquad \mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{-1}[\mathbf{A}^f\mathbf{k} + \mathbf{b}l(z^{-1})]\frac{dz}{z}.\end{aligned}$$

With this result, we can write the performance index as

$$\begin{aligned}\sigma^2 &= \sigma_y^2 + \lambda\sigma_u^2,\\ &= \sigma_{y,mv}^2 + \frac{\sigma_a^2}{2\pi i}\oint_C \mathbf{c}[\mathbf{I}-\mathbf{A}z]^{-1}[\mathbf{A}^f\mathbf{k}+\mathbf{b}l(z)]\mathbf{c}[\mathbf{I}-\mathbf{A}z^{-1}]^{-1}\\ &\qquad [\mathbf{A}^f\mathbf{k} + \mathbf{b}l(z^{-1})]\frac{dz}{z} + \frac{\sigma_a^2}{2\pi i}\oint_C \lambda l(z)l(z^{-1})\frac{dz}{z},\\ &= \sigma_{y,mv}^2 + \frac{\sigma_a^2}{2\pi i}\oint_C \frac{\mathbf{c}[\mathbf{I} - \mathbf{A}z]^{+}[\mathbf{A}^f\mathbf{k} + \mathbf{b}l(z)]}{|\mathbf{I} - \mathbf{A}z|} \times\\ &\qquad \frac{\mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{+}[\mathbf{A}^f\mathbf{k} + \mathbf{b}l(z^{-1})]}{|\mathbf{I} - \mathbf{A}z^{-1}|z}dz\\ &\quad + \frac{\sigma_a^2}{2\pi i}\oint \frac{\lambda l(z)|\mathbf{I} - \mathbf{A}z|\,|\mathbf{I} - \mathbf{A}z^{-1}|l(z^{-1})}{|\mathbf{I} - \mathbf{A}z|\,|\mathbf{I} - \mathbf{A}z^{-1}|z}dz.\end{aligned}$$

The control problem has been solved in K. Vu (2008) and the open loop controller that minimizes the performance index is

$$l(z^{-1}) = -\frac{\beta(z^{-1})}{\alpha(z^{-1})}$$

with the polynomials $\alpha(z^{-1})$ and $\beta(z^{-1})$ satisfy the following equations.

$$\begin{aligned}\alpha(z)\alpha(z^{-1}) &= \mathbf{c}[\mathbf{I}-\mathbf{A}z]^{+}\mathbf{b}\mathbf{c}[\mathbf{I}-\mathbf{A}z^{-1}]^{+}\mathbf{b}\\ &\qquad +\lambda|\mathbf{I} - \mathbf{A}z|\,|\mathbf{I} - \mathbf{A}z^{-1}|,\\ \frac{\mathbf{c}[\mathbf{I} - \mathbf{A}z]^{+}\mathbf{A}^f\mathbf{k}\mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{+}\mathbf{b}}{|\mathbf{I} - \mathbf{A}z|\alpha(z^{-1})} &= \frac{\beta(z)}{|\mathbf{I} - \mathbf{A}z|} + \frac{\zeta(z^{-1})}{\alpha(z^{-1})}z^{-1}.\end{aligned}$$

The first equation is called a spectral factorization equation; the second equation, a spectral separation equation. The minimum performance index has the value

$$\begin{aligned}\hat{\sigma}^2 &= \sigma_{y,mv}^2 + \lambda\frac{\sigma_a^2}{2\pi i}\oint_C \frac{\mathbf{c}[\mathbf{I} - \mathbf{A}z]^{+}\mathbf{A}^f\mathbf{k}\mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{+}\mathbf{A}^f\mathbf{k}}{\alpha(z)\alpha(z^{-1})z}dz\\ &\quad + \frac{\sigma_a^2}{2\pi i}\oint_C \frac{\zeta(z)\zeta(z^{-1})}{\alpha(z)\alpha(z^{-1})z}dz.\end{aligned}$$

To obtain the controller in terms of the input and output variables, we write

$$\begin{aligned}u_t &= -\frac{\beta(z^{-1})}{\alpha(z^{-1})}a_t,\\ y_t &= (1 + \mathbf{c}[\mathbf{I} - \mathbf{A}z^{-1}]^{-1}[-\mathbf{b}\frac{\beta(z^{-1})}{\alpha(z^{-1})}z^{-f} + \mathbf{k}]z^{-1})a_t,\\ &= (1 + \mathbf{c}\frac{[\mathbf{I} - \mathbf{A}z^{-1}]^{+}}{|\mathbf{I} - \mathbf{A}z^{-1}|}[-\mathbf{b}\frac{\beta(z^{-1})}{\alpha(z^{-1})}z^{-f} + \mathbf{k}]z^{-1})a_t.\end{aligned}$$

By eliminating the variable a_t from these time series equations, we can obtain the feedback controller as

$$u_t = \frac{-|\mathbf{I}-\mathbf{A}z^{-1}|\beta(z^{-1})}{|\mathbf{I}-\mathbf{A}z^{-1}|\alpha(z^{-1})+\mathbf{c}[\mathbf{I}-\mathbf{A}z^{-1}]^{+}\mathbf{k}\alpha(z^{-1})z^{-1} - \mathbf{c}[\mathbf{I}-\mathbf{A}z^{-1}]^{+}\mathbf{b}\beta(z^{-1})z^{-f-1}} y_t.$$

3 An Example

The verification of the infinite steps optimal control algorithm is an easy chore. All one has to do is to calculate the variances of the input and output variables and compare their weighted sum with the performance index value of the control algorithm. Therefore, we will leave this task as an exercise to the enterprising readers. We will consider an example to verify the one step optimal control algorithm because other control algorithms depend on this one.

Suppose that we have a control system with the following data:

$$\mathbf{A}=\begin{bmatrix} 0.6945 & 1.7235 \\ -0.3041 & -0.5659 \end{bmatrix}, \mathbf{b}=\begin{bmatrix} 0.8968 \\ 0.1379 \end{bmatrix}, \mathbf{c}=\begin{bmatrix} 0.6166 & 0.1133 \end{bmatrix}$$

with the pure dead time $f=1$ and the noises statistics:

$$\mathbf{R}_w=\begin{bmatrix} 2.8054 & 2.3500 \\ 2.3500 & 3.2548 \end{bmatrix}, \sigma_v^2=0.3191.$$

The performance index of the one step optimal control algorithm is a conditional expectation. A conditional expectation will give us the numerical values of a variable, not the variance of a time series. Therefore, to be able to verify the performance index, we have to assume that we have an infinite amount of data; this will change a conditional expectation to an unconditional expectation. The assumption is acceptable because it is applied to both sides of the equation for the performance index. It will not change the truth of the equation. With this established, we can use a time series variance formula to calculate the variances of the variables for verification purpose.

With the mentioned assumption, we have the modified performance index as

$$\begin{aligned} Min\ \ E\{y_{t+f+1}^2\}+\lambda E\{u_t^2\} &= tr(\mathbf{A}^{f+1^T}\mathbf{c}^T\mathbf{c}\mathbf{A}^{f+1}\mathbf{P}_{t|t}) \\ &+tr(\sum_{i=0}^{f}\mathbf{A}^{i^T}\mathbf{c}^T\mathbf{c}\mathbf{A}^{i}\mathbf{R}_w)+\sigma_v^2 \\ +tr(\left[\mathbf{A}^T\mathbf{c}^T\mathbf{c}\mathbf{A}-\frac{\mathbf{A}^T\mathbf{c}^T\mathbf{c}\mathbf{b}\mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{A}}{\lambda+\mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{b}}\right] & E\{\hat{\mathbf{x}}_{t+f|t}\hat{\mathbf{x}}_{t+f|t}^T\}). \end{aligned}$$

To verify this equation, we have to find the equations of the time series y_t, u_t and $\hat{\mathbf{x}}_{t+f|t}$. The time series y_t and u_t are scalar ARMA time series, but the time series $\hat{\mathbf{x}}_{t+f|t}$ is a VARMA time series. We can, however, find all the formulae to calculate the variances or moments of these time series in K. Vu (2007). With the pure dead time $f=1$, we have

$$u_t = -\mathbf{l}\hat{\mathbf{x}}_{t+1|t}$$

and the one step ahead state estimator given recursively as

$$\begin{aligned} \hat{\mathbf{x}}_{t+1|t} &= \mathbf{A}\hat{\mathbf{x}}_{t|t-1}+\mathbf{b}u_{t-1}+\mathbf{k}a_t, \\ \hat{\mathbf{x}}_{t+1|t} &= \mathbf{A}\hat{\mathbf{x}}_{t|t-1}-\mathbf{b}\mathbf{l}\hat{\mathbf{x}}_{t|t-1}+\mathbf{k}a_t, \\ &= [\mathbf{A}-\mathbf{b}\mathbf{l}]\hat{\mathbf{x}}_{t|t-1}+\mathbf{k}a_t. \end{aligned}$$

The state estimator is a VAR(1) time series. From the above equation, we can write

$$\hat{\mathbf{x}}_{t+1|t} = [\mathbf{I}-(\mathbf{A}-\mathbf{b}\mathbf{l})z^{-1}]^{-1}\mathbf{k}a_t.$$

Therefore, the time series for the input variable u_t is

$$|[\mathbf{I}-(\mathbf{A}-\mathbf{b}\mathbf{l})z^{-1}]|u_t = -\mathbf{l}[\mathbf{I}-(\mathbf{A}-\mathbf{b}\mathbf{l})z^{-1}]^{+}\mathbf{k}a_t.$$

From the reconstructed model, the output variable is given as

$$y_t = \mathbf{c}\hat{\mathbf{x}}_{t|t-1}+a_t.$$

Therefore, the time series for the output variable y_t is

$$y_t = \frac{[|\mathbf{I}-(\mathbf{A}-\mathbf{b}\mathbf{l})z^{-1}|+\mathbf{c}[\mathbf{I}-(\mathbf{A}-\mathbf{b}\mathbf{l})z^{-1}]^{+}\mathbf{k}z^{-1}]}{|\mathbf{I}-(\mathbf{A}-\mathbf{b}\mathbf{l})z^{-1}|}a_t.$$

Now, with the given parameters of the model, we can obtain the following values. With the penalty constant $\lambda=0.05$, we have

$$\begin{aligned} \mathbf{l} &= [\lambda+\mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{b}]^{-1}\mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{A}, \\ &= \begin{bmatrix} 0.5997 & 1.5210 \end{bmatrix}, \\ \mathbf{A}-\mathbf{b}\mathbf{l} &= \begin{bmatrix} 0.1566 & 0.3595 \\ -0.3868 & -0.7756 \end{bmatrix}. \end{aligned}$$

To proceed further, we must obtain the Kalman filter and the one step ahead state estimator. Solving the Riccati equation, we have

$$\begin{aligned} \mathbf{P}_{t+1|t} &= \mathbf{A}\mathbf{P}_{t+1|t}\mathbf{A}^T-\frac{\mathbf{A}\mathbf{P}_{t+1|t}\mathbf{c}^T\mathbf{c}\mathbf{P}_{t+1|t}\mathbf{A}^T}{\sigma_v^2+\mathbf{c}\mathbf{P}_{t+1|t}\mathbf{c}^T}+\mathbf{R}_{\mathbf{w}}, \\ &= \begin{bmatrix} 14.4536 & -1.4131 \\ -1.4131 & 4.4758 \end{bmatrix}, \\ \mathbf{k} &= \frac{\mathbf{A}\mathbf{P}_{t+1|t}\mathbf{c}^T}{\sigma_v^2+\mathbf{c}\mathbf{P}_{t+1|t}\mathbf{c}^T} = \begin{bmatrix} 0.9605 \\ -0.4327 \end{bmatrix}, \\ \mathbf{P}_{t|t} &= \mathbf{P}_{t+1|t}-\mathbf{P}_{t+1|t}\mathbf{c}^T(\sigma_v^2+\mathbf{c}\mathbf{P}_{t+1|t}\mathbf{c}^T)^{-1}\mathbf{c}\mathbf{P}_{t+1|t}, \\ &= \begin{bmatrix} 0.9548 & -0.8516 \\ -0.8516 & 4.4524 \end{bmatrix}, \\ \sigma_a^2 &= \sigma_v^2+\mathbf{c}\mathbf{P}_{t+1|t}\mathbf{c}^T, \\ &= 5.6747. \end{aligned}$$

For the time series, we can obtain

$$\begin{aligned} (1+0.619z^{-1}+0.0176z^{-2})u_t &= (1+1.3214z^{-1})a_t, \\ \sigma_u^2 &= 0.0691, \\ (1+0.619z^{-1}+0.0176z^{-2})y_t &= (1+1.1622z^{-1}+0.3466z^{-2})a_t, \\ \sigma_y^2 &= 7.3498. \end{aligned}$$

The state estimator $\hat{\mathbf{x}}_{t+1|t}$ is a VAR(1) time series with the parameters:

$$\begin{aligned} \boldsymbol{\vartheta}(z^{-1}) &= \mathbf{I}, \\ \boldsymbol{\varphi}(z^{-1}) &= (\mathbf{I} - \begin{bmatrix} 0.1566 & 0.3595 \\ -0.3868 & -0.7756 \end{bmatrix} z^{-1}), \\ \mathbf{R}_a = \sigma_a^2 \mathbf{k}\mathbf{k}^T &= \begin{bmatrix} 5.2352 & -2.3584 \\ -2.3584 & 1.0624 \end{bmatrix}. \end{aligned}$$

With this information, we can proceed to obtain its variance matrix as

$$E\{\hat{\mathbf{x}}_{t+1|t}\hat{\mathbf{x}}_{t+1|t}^T\} = \begin{bmatrix} 5.2370 & -2.3610 \\ -2.3610 & 1.0775 \end{bmatrix}.$$

For the performance index, we can obtain

$$\begin{aligned} \sigma_y^2 + \lambda\sigma_u^2 &= 7.3498 + 0.05 \times 0.0691, \\ &= 7.3532 \end{aligned}$$

and the following variances:

$$\begin{aligned} tr(\mathbf{A}^{f+1^T}\mathbf{c}^T\mathbf{c}\mathbf{A}^{f+1}\mathbf{P}_{t|t}) &= 0.0641, \\ tr(\sum_{i=0}^{f}\mathbf{A}^{i^T}\mathbf{c}^T\mathbf{c}\mathbf{A}^i\mathbf{R}_w) &= 6.9660, \\ \sigma_v^2 &= 0.3191, \\ tr(\left[\mathbf{A}^T\mathbf{c}^T\mathbf{c}\mathbf{A} - \frac{\mathbf{A}^T\mathbf{c}^T\mathbf{c}\mathbf{b}\mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{A}}{\lambda + \mathbf{b}^T\mathbf{c}^T\mathbf{c}\mathbf{b}}\right]\Gamma_{\hat{\mathbf{x}}\hat{\mathbf{x}}}(0)) &= 0.0040, \\ \sum &= 7.3532. \end{aligned}$$

With the above results, we can say that the performance index of the one step optimal control algorithm is correct.

4 Conclusion

In this paper, we have presented four optimal discrete controllers for control applications. The one step optimal controller is the simplest controller. It shows a clear relation between the input and output variables and the control penalty constant under feedback. The N steps optimal control algorithm can only apply when we know the exact number of control steps of the control algorithm. The pseudo infinite steps optimal controller is a cross controller between the N steps optimal control algorithm and the infinite steps optimal controller. It can be used as an easy extension to multivariable systems in place of the infinite steps optimal controller. For SISO systems, the infinite steps optimal controller, a research result of AuLac Technologies Inc., is the most preferable control algorithm.

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