Discrete H_∞ Control Theory for Chemical Process Control Engineers

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Abstract

In this paper, the discrete $\mathbf{H}_\infty$ least sensitivity controller, which minimizes the infinity norm of a weighted sensitivity function is presented. The minimal weighted sensitivity function, which can be interpreted as a Box-Jenkins model for a stochastic regulating control system with the ARIMA time series disturbance model as the weighting function, has a flat spectrum. The controller is obtained by searching for an embedding polynomial and solving for the minimum variance controller. When there is no pure dead time, the least sensitivity controller is the same as the minimum variance controller of the Box-Jenkins model.

Keywords: ARIMA time series, Box-Jenkins model, $\mathbf{H}_\infty$ control, sensitivity functions.

1 Introduction

Chemical process control engineers are a big group of control engineers. This is because, by nature, the chemical industries are usually big industries. Big industries such as the oil and gas and pulp and paper employ a good number of process control engineers. While the industries use advanced process control theory to control their processes, $\mathbf{H}_\infty$ control theory is not well known among chemical process control engineers. Many chemical process control engineers believe $\mathbf{H}_\infty$ control theory is frequency domain control theory, which then should be more suited to electrical control engineers. This is a misconception, which should be rectified and chemical process control engineers can benefit from this relatively recent development in control theory. In this paper, a brief description of discrete $\mathbf{H}_\infty$ control theory is presented and a method to obtain the least sensitivity controller is discussed. The paper is organized as follows. Section one is the introduction section. Section two describes the system norms. In section three, the sensitivity functions are listed. In section four, the least sensitivity controller is derived. An example is considered in section five and section six concludes the discussion of the paper.

2 Signal and System Norms

For the purpose of measuring the size of a signal spectrum, a norm is defined. The L_n-norm of a discrete signal x_t with its Fourier transform $x(e^{-i\omega})$ is defined as

$$\parallel x(e^{-i\omega}) \parallel_n = \sqrt[n]{\frac{1}{2\pi}\int_{-\pi}^{\pi} |x(e^{-i\omega})|^n d\omega}.$$

The L_1-norm is the average absolute value of the signal. It can be determined by either numerical integration of the magnitude function or by computing the Discrete Fourier Transform of the impulse response coefficients.

The L_2-norm is the root mean squared value of the power contained in the signal. The L_2-norm of a discrete signal has a similar interpretation. The L_2-norm is given as

$$\parallel x(e^{-i\omega}) \parallel_2 = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |x(e^{-i\omega})|^2 d\omega}.$$

The L_2-norm is simple to calculate, and it can be calculated via an integration in the frequency domain or a summation in the time domain. The Parseval's relation allows us to calculate it in either domain. In a stochastic environment, the L_2-norm of a zero mean signal is known as its standard deviation.

It can be proved that if n is higher, the L_n-norm is higher. Therefore, one is interested in the norm corresponding to the highest possible value of n, which is infinity. The L_∞-norm corresponds to the maximum value of the magnitude function, since we have

$$\begin{aligned}\parallel x(e^{-i\omega}) \parallel_\infty &= \lim_{n\to\infty} \parallel x(e^{-i\omega}) \parallel_n, \\ &= \lim_{n\to\infty} \sqrt[n]{\frac{1}{2\pi}\int_{-\pi}^{\pi} |x(e^{-i\omega})|^n d\omega}, \\ &= \sup_{\omega} |x(e^{-i\omega})|.\end{aligned}$$

The H_∞ controller uses this norm, and the linear quadratic controller uses the L_2-norm. The maximum gain of an amplifier over a frequency band and the resonant gain of an electric filter correspond to the infinity norm.

A signal is usually the output variable of a linear system or filter with some input variable; therefore, we can infer the size of the signal from the size of the transfer function and the size of the input variable. The size of the signal will be the same as the size of the transfer function if the size of the input variable has unit value.

The L_1-norm of a system is the most complicated norm to calculate and the least interested. It can be calculated as mentioned above. The L_2-norm is more interested, and it can be calculated by residue calculus. For the transfer function $G_p(z^{-1})$, the L_2-norm or quadratic norm is given as

$$\| G_p(z^{-1}) \|_2 \quad = \quad \sqrt{\underset{z=0}{Residue}\ G_p(z)G_p(z^{-1})\frac{1}{z}}.$$

The L_∞-norm can be deduced from the spectral function $G_p(e^{i\omega})G_p(e^{-i\omega})$. Since it is the supremum, it exists as the maximum of this function or one of the ordinates at the two end points of the spectrum, i.e. at the zero or Nyquist frequency. Reference K. Vu (2007) gives an equation that can be used to calculate the infinity norm of a system with a rational transfer function.

3 Sensitivity Functions

A general feedback control system with both set point and load disturbances can be described by the following Fig. 1.

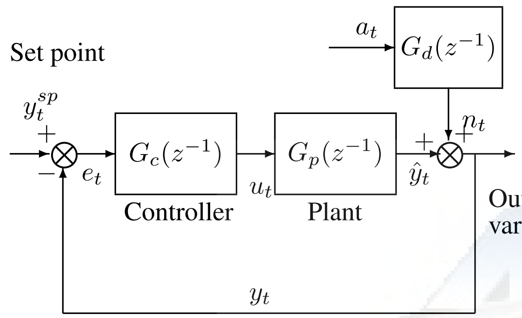

Figure 1. A Feedback Control System.

From the block diagram of this figure, we can write

$$\begin{aligned} y_t &= \hat{y}_t + n_t, \\ &= G_p(z^{-1})u_t + G_d(z^{-1})a_t, \\ &= G_p(z^{-1})G_c(z^{-1})e_t + G_d(z^{-1})a_t, \\ &= G_p(z^{-1})G_c(z^{-1})[y_t^{sp} - y_t] + G_d(z^{-1})a_t. \end{aligned}$$

By bringing the term with the variable y_t from the right hand side of the above equation to the left hand side, we can write

$$\begin{aligned} [1 + G_p(z^{-1})G_c(z^{-1})]y_t &= G_p(z^{-1})G_c(z^{-1})y_t^{sp} \\ &\quad + G_d(z^{-1})a_t. \end{aligned}$$

Therefore, the equation describing the system is given as

$$y_t = \frac{G_p(z^{-1})G_c(z^{-1})}{1 + G_p(z^{-1})G_c(z^{-1})}y_t^{sp} + \frac{G_d(z^{-1})}{1 + G_p(z^{-1})G_c(z^{-1})}a_t.$$

For either tracking or regulating control, we have a common polynomial as a pole polynomial in the transfer functions with the disturbances as the input variables. This polynomial determines the stability of the system.

The pivotal idea of H_∞ control is a function called the *sensitivity function* that is the inverse of the aforementioned stability polynomial. This means that the sensitivity function was defined as

$$S \quad = \quad \frac{1}{1 + G_p(z^{-1})G_c(z^{-1})}.$$

For regulating control, S is the transfer function from n_t to y_t. From this sensitivity function, a couple of other functions were also defined as shown below. We have

$$\begin{aligned} U &= \text{Control Sensitivity}, \\ &= \text{Input Sensitivity}, \\ &= \frac{G_c(z^{-1})}{1 + G_p(z^{-1})G_c(z^{-1})}, \\ &= G_c(z^{-1})S. \end{aligned}$$

The control or input sensitivity function is the product of the controller and the sensitivity function. It is the negative of the transfer function from the disturbance n_t to the plant input variable u_t. Similarly, we have

$$\begin{aligned} T &= \text{Complementary Sensitivity}, \\ &= 1 - S, \\ &= 1 - \frac{1}{1 + G_p(z^{-1})G_c(z^{-1})}, \\ &= \frac{G_p(z^{-1})G_c(z^{-1})}{1 + G_p(z^{-1})G_c(z^{-1})}, \\ &= G_p(z^{-1})U. \end{aligned}$$

The complementary sensitivity function is the product of the control sensitivity function and the plant model. The sensitivity functions are used in a performance index equation to design the H_∞ controllers.

4 Least Sensitivity Controller

The performance index for the controller is

$$\underset{G_c(z^{-1})}{Min} \| VS \|_\infty \quad = \quad \underset{G_c(z^{-1})}{Min} \sup \| VS \|_2 \ .$$

The frequency weighting function V has a dual meaning. A control engineer would like to think that this is his design choice, but another meaning that is more meaningful is that the function is the natural frequency weighting of the disturbance.

$$\begin{aligned} V &= G_d(z^{-1}), \\ &= \frac{\theta(z^{-1})}{\phi(z^{-1})} \end{aligned}$$

and

$$VS \quad = \quad \frac{G_d(z^{-1})}{1 + G_p(z^{-1})G_c(z^{-1})}.$$

We can now propose an approach to obtain the solution by writing the output variable of the control system as

$$\begin{aligned} y_t &= \frac{\omega(z^{-1})}{\delta(z^{-1})}z^{-f-1}u_t + G_d(z^{-1})a_t, \\ &= \frac{\omega(z^{-1})}{\delta(z^{-1})}z^{-f-1}u_t + \frac{\theta(z^{-1})}{\phi(z^{-1})}a_t. \end{aligned}$$

The above model is, of course, the Box-Jenkins model described in G. Box and G. Jenkins (1976) because the control problem is a regulating control problem. We can suggest that the controller equation is of the following form

$$u_t = l(z^{-1})a_t.$$

It is necessary to express and derive the controller in this form because the source of the disturbance is the variable a_t. With the controller in this form, we can get the right number of the controller parameters and their optimal values. With the above equations for the input and output variables, we can say that when the loop is closed, we have the following equation

$$\begin{aligned}
y_t &= \frac{\omega(z^{-1})}{\delta(z^{-1})} z^{-f-1} u_t + G_d(z^{-1}) a_t, \\
&= \frac{\omega(z^{-1})}{\delta(z^{-1})} z^{-f-1} l(z^{-1}) a_t + \frac{\theta(z^{-1})}{\phi(z^{-1})} a_t, \\
&= \left(\frac{\omega(z^{-1})}{\delta(z^{-1})} z^{-f-1} l(z^{-1}) + \frac{\theta(z^{-1})}{\phi(z^{-1})} \right) a_t, \\
&= \frac{\theta(z^{-1})\delta(z^{-1}) + \omega(z^{-1})\phi(z^{-1}) l(z^{-1}) z^{-f-1}}{\delta(z^{-1})\phi(z^{-1})} a_t, \\
&= \frac{G_d(z^{-1})}{1 + G_p(z^{-1}) G_c(z^{-1})} a_t.
\end{aligned}$$

Therefore, we can write the performance index for the controller as

$$\underset{G_c(z^{-1})}{Min} \parallel VS \parallel_\infty = \underset{G_c(z^{-1})}{Min} \sup \parallel VS \parallel_2$$

or

$$\underset{G_c(z^{-1})}{Min} \sup \parallel \frac{G_d(z^{-1})}{1 + G_p(z^{-1}) G_c(z^{-1})} \parallel_2$$

and

$$\underset{l(z^{-1})}{Min} \sup \parallel \frac{\theta(z^{-1})}{\phi(z^{-1})} + \frac{\omega(z^{-1}) l(z^{-1}) z^{-f-1}}{\delta(z^{-1})} \parallel_2 .$$

The solution for this problem has been solved in K. Vu (2008). The solution of the problem is

$$l(z^{-1}) = -\frac{\delta(z^{-1})\gamma_a(z^{-1})}{\omega(z^{-1})\phi(z^{-1}) d_1(z^{-1})}$$

with the new polynomials satisfying the equations:

$$\frac{d_1(z^{-1})\theta(z^{-1})}{d_2(z^{-1})\phi(z^{-1})} = \psi_a(z^{-1}) + \frac{\gamma_a(z^{-1})}{d_2(z^{-1})\phi(z^{-1})} z^{-f-1}$$

and

$$\left[\frac{\psi_a(z^{-1}) d_2(z^{-1})}{d_1(z^{-1})} \right] \left[\frac{\psi_a(z) d_2(z)}{d_1(z)} \right] = \mu.$$

Under feedback with this controller, the output variable follows the time series

$$y_t = \frac{\psi_a(z^{-1})}{d_1(z^{-1})} a_t,$$

which has a flat spectrum. The closed-loop controller can be derived to have the following form

$$\begin{aligned}
G_c(z^{-1}) &= -\frac{u(z^{-1})}{y(z^{-1})}, \\
&= \frac{\delta(z^{-1})\gamma_a(z^{-1})}{\omega(z^{-1})\phi(z^{-1})\psi_a(z^{-1})}.
\end{aligned}$$

The feedback system has the following sensitivity functions:

$$\begin{aligned}
S &= \frac{\phi(z^{-1})\psi_a(z^{-1})}{\theta(z^{-1}) d_1(z^{-1})}, \\
U &= \frac{\delta(z^{-1})\gamma_a(z^{-1})}{\omega(z^{-1})\theta(z^{-1}) d_1(z^{-1})}, \\
T &= \frac{\gamma_a(z^{-1}) z^{-f-1}}{\theta(z^{-1}) d_1(z^{-1})}.
\end{aligned}$$

The least sensitivity controller gives the output variable y_t a flat spectrum. When $f = 0$, the minimum variance of a Box-Jenkins model gives the output variable a white noise with a flat spectrum. Therefore, for this case, the least sensitivity controller is the same as the minimum variance controller and we have $\psi_a(z^{-1}) = d_1(z^{-1}) = 1$.

A flat spectrum is not always a desirable spectrum for the output variable. And we have said that the frequency weighting model $G_d(z^{-1})$ is not likely a control engineer's choice but a natural frequency distribution of the disturbance. This is because the disturbance seldom comes in the system as a white noise. Therefore, instead of looking for the minimax of the spectrum of the function

$$VS = \frac{G_d(z^{-1})}{1 + G_p(z^{-1}) G_c(z^{-1})},$$

we look for the minimax of the spectrum of the function

$$VS = \frac{G_w(z^{-1}) G_d(z^{-1})}{1 + G_p(z^{-1}) G_c(z^{-1})}$$

where $G_w(z^{-1})$ is the control engineer's choice of a frequency weighting model. With the above choice of the weighted frequency sensitivity function, we have the spectrum of VS flat but the spectrum of the output variable y_t given by the transfer function

$$y_t = \overbrace{\frac{1}{G_w(z^{-1})}}^{Certainty\ Part} \overbrace{\frac{\psi_a(z^{-1})}{d_1(z^{-1})}}^{Uncertainty\ Part} a_t$$

a desirable curve.

5 An Example

In this section, we consider a numerical example. The design of the least sensitivity controller is accomplished by the function **hinfcon** of the RTF-SISO Discrete Control Toolbox (2008), which runs in Matlab[1] software.

[1]Matlab is a registered trademark of The MathWorks, Inc.

The models of the control system:

$$
\begin{aligned}
G_p(z^{-1}) &= \frac{3.6051 - 2.1266z^{-1}}{1 - 0.1942z^{-1} - 0.2144z^{-2}} z^{-2}, \\
G_d(z^{-1}) &= \frac{1 - 0.2665z^{-1}}{1 + 0.1388z^{-1}}.
\end{aligned}
$$

are generated by the function **crbjcs** of the same software package. The frequency weighting model, the engineer's choice, for the example is $G_w(z^{-1}) = 1 - 0.65z^{-1}$. With these values, the function **hinfcon** of the RTF-SISO Discrete Control Toolbox (2008) returns the following polynomials:

$$
\begin{aligned}
d_1(z^{-1}) &= 1 - 0.6030z^{-1}, \\
\psi_a(z^{-1}) &= 1 - 1.6583z^{-1}, \\
\gamma_a(z^{-1}) &= 0.9560 - 0.1044z^{-1}
\end{aligned}
$$

with the following sensitivity functions:

$$
\begin{aligned}
S &= \frac{1 - 1.5195z^{-1} - 0.2302z^{-2}}{1 - 1.5195z^{-1} + 0.7259z^{-2} - 0.1044z^{-3}}, \\
U &= \frac{0.9560 - 0.2901z^{-1} - 0.1847z^{-2} + 0.0224z^{-3}}{3.6051 - 3.8002z^{-1} + 0.6855z^{-2} + 0.1780z^{-3}}, \\
T &= \frac{0.9560z^{-2} - 0.1044z^{-3}}{1 - 1.5195z^{-1} + 0.7259z^{-2} - 0.1044z^{-3}}
\end{aligned}
$$

and their weighted ones as

$$
\begin{aligned}
G_d S &= \frac{1 - 1.6583z^{-1}}{1 - 1.2530z^{-1} + 0.3920z^{-2}}, \\
G_w G_d S &= \frac{1 - 1.6583z^{-1}}{1 - 0.6030z^{-1}}.
\end{aligned}
$$

The closed-loop controller for the control system has the form

$$
u_t = \frac{-0.2652 + 0.0805z^{-1} + 0.0512z^{-2} - 0.0062z^{-3}}{1 - 2.1094z^{-1} + 0.6662z^{-2} + 0.1358z^{-3}} y_t,
$$

and it gives the performance index the value

$$
\mu = 2.7499.
$$

Because of the choice $G_w(z^{-1}) \neq 1$, the spectrum of $G_w G_d S$ is flat with the value μ but that of $G_d S$ is not. The function **hinfcon** has an option to plot the spectra of all the sensitivity functions. But these spectra are only power spectra. For other spectra, for example phase spectra, control engineers can use functions in the V-ARIMA Time series Toolbox (2008).

6 Conclusion

In this paper, we have discussed the $\mathbf{H}_\infty$ least sensitivity controller. The least sensitivity controller is usually discussed in a stochastic environment. There are a number of stochastic controllers. Reference K. Vu (2008a) classifies them according to the number of steps in the performance index. The controller hence can be nominated with the number of steps of optimality. The least sensitivity controller, being a frequency controller, is an infinite steps optimal controller. However, unlike the infinite steps optimal controller with the performance index as a weighted sum of two variances, the least sensitivity controller can shape the spectrum of the output variable. It has greater control and flexibility over the spectrum of the output variable. It is, therefore, a more desirable controller than other stochastic controllers.

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